The Apple Doesn't Fall Far From the (Metric) Tree: Equivalence of Definitions

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Abstract. In this paper we prove the equivalence of definitions for metric trees and for δ-Hperbolic spaces. We point out how these equivalences can be used to understand the geometric and metric properties of δ-Hperbolic spaces and its relation to CAT(κ) spaces.

1 Introduction

A metric space is a metric tree if and only if it is 0-hyperbolic and geodesic. In other words, a geodesic metric space is said to be a metric tree (or an $\mathbb{R}$-tree, or T-tree) if it is 0-hyperbolic in the sense of Gromov that all of its geodesic triangles are isometric to tripods. It is well known that 0-hyperbolic metric space embeds isometrically into a metric tree (see [14],[20]) and construction of metric trees relate to the asymptotic geometry of hyperbolic spaces (see [11], [17]). Metric trees are not only described by different names but also they are given by different definitions, and in the following we state two widely used definitions of a metric tree:

Definition 1.1. An $\mathbb{R}$-tree is a metric space M such that for every x and y in M there is a unique arc between x and y and this arc is isometric to an interval in $\mathbb{R}$ (i.e., is a geodesic segment).

Recall that for $x, y \in M$ a *geodesic segment* from x to y denoted by $[x, y]$ and is the image of an isometric embedding $\alpha : [a, b] \to M$ such that $\alpha(a) = x$ and $\alpha(b) = y$. A geodesic metric space is a metric space in which every pair of points is joined by a (not necessarily unique) geodesics.

Definition 1.2. An $\mathbb{R}$-tree is a metric space M such that

(i) there is a unique geodesic segment denoted by $[x, y]$ joining each pair of points x and y in M;

(ii) if $[y, x] \cap [x, z] = \{x\} \Rightarrow [y, x] \cup [x, z] = [y, z]$.

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Condition (ii) above simply states that if two segments intersect in a single point then their union is a segment too. Note that $\mathbb{R}^n$ with the Euclidean metric satisfies the first condition. It fails, however, to satisfy the second condition.
The study of metric trees is motivated by many subdisciplines of mathematics [18], [34], biology/medicine and computer science. The relationship between metric trees and biology and medicine stems from the construction of phylogenetic trees [33]; and concepts of "string matching" in computer science are closely related with the structure of metric trees [6]. Unlike metric trees, in an ordinary tree all the edges are assumed to have the same length and therefore the metric is not often stressed. However, a metric tree is a generalization of an ordinary tree that allows for different edge lengths. For example, a connected graph without loops is a metric tree. Metric trees also arise naturally in the study of group isometries of hyperbolic spaces. For metric properties of trees we refer to [13]. Lastly, [31] and [32] explore topological characterization of metric trees. For an overview of geometry, topology, and group theory applications of metric trees,consult Bestvina [7]. For a complete discussion of these spaces and their relation to $CAT(\kappa)$ spaces we refer to[11]. If the metric d is understood, we will denote $d(x, y)$ by xy. We also say that a point z is *between* x and y if $xy = xz + zy$. We will often denote this by xzy. It is not difficult to prove that in any metric space, the elements of a metric segment from x to y are necessarily between x and y, and in a metric tree, the elements between x and y are the elements in the unique metric segment from x to y. Hence, if M is a metric tree and $x, y \in M$, then

$$[x, y] = \{z \in M : xy = xz + zy\}.$$

The following is an example of a metric tree. For more examples see [3].

Example 1.1. *(The Radial Metric) Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ by:*

$$d(x, y) = \begin{cases} \|x - y\| & \text{if } x = \lambda\, y \text{ for some } \lambda \in I\!R, \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

We can observe that the d is in fact a metric and that $(\mathbb{R}^2, d)$ is a metric tree.

It is well known that any complete, simply connected Riemannian manifold having non-positive curvature is a $CAT(0)$-space. Other examples include the complex Hilbert ball with the hyperbolic metric (see [21]), Euclidean buildings (see [12]) and classical hyperbolic spaces. If a space is $CAT(\kappa)$ for some $\kappa < 0$ then it is automatically $CAT(0)$-space. Metric trees, is a sub-class of $CAT(0)$-spaces, also it is useful to mention the following:

Proposition 1.2. *If a metric space is $CAT(\kappa)$ space for all κ then it is a metric tree.*

For the proof of the above proposition we refer to [11]. Note that if a Banach space is a $CAT(\kappa)$ space for some κ then it is necessarily a Hilbert space and $CAT(0)$. The property that distinguishes the metric trees from the $CAT(0)$ spaces is the fact that metric trees are hyperconvex metric spaces. Properties of hyperconvex spaces and their relation to metric trees can be found in [1], [5], [25] and [27]. We refer to [8] for the properties of metric segments and to [2] and [4] for the basic properties of complete metric trees. In the following we list some of the properties of metric trees which will be used in the proof of Theorem 2.1.

1. (Uniform Convexity [3]). A metric tree M is uniformly convex.
2. (Projections are nonexpansive [4]). Metric projections on closed convex subsets of a metric trees are nonexpansive.

Property 1 above generalizes the classical Banach space notion of uniform convexity by defining modulus of convexity for geodesic metric spaces. Let C be a closed convex subset (by convex we mean for all $x, y \in C$, we have $[x, y] \subset C$) of a metric tree M. If for every point $x \in M$ there exists a nearest point in C to x, and if this point is unique, we denote this point by $P_C(x)$, and call the mapping P_C the metric projection from M into C. In Hilbert space, the metric projections on closed convex subsets are nonexpansive. In uniformly convex spaces, the metric projections are uniformly Lipschitzian. In fact, they are nonexpansive if and only if the space is Hilbert. Property 2 is remarkable in this context and this result is not known in hyperconvex spaces. However, the fact that the nearest point projection onto convex subsets of metric trees is nonexpsansive also follows from the fact that this is true in the more general setting of $CAT(0)$ spaces (see p. 177 of [11]).
A metric space (X, d) is said to have *four-point property* if for each $x, y, z, p \in X$

$$d(x,y) + d(z,p) \le \max\{d(x,z) + d(y,p), d(x,p) + d(y,z)\}$$

holds. Four-point property characterizes metric trees [1] thus, it is natural extension characterizes δ-hyperbolic spaces as seen in Definition1.4 below.
In the following we now give three widely used definition δ-hyperbolic spaces and references to how these definitions are utilized in order to describe geometric properties.

Definition 1.3. A metric space (X, d) is δ-hyperbolic if for all $p, x, y, z \in X$,

$$(x,z)_p \ge \min\left\{(x,y)_p, (y,z)_p\right\} - \delta \tag{1.1}$$

where $(x,z)_p = \frac{1}{2}(d(x,p) + d(z,p) - d(x,z))$ is the Gromov product.

Definition 1.4. A metric space (X, d) is called $\delta - hyperbolic$ for $\delta \geq 0$ if for each $x, y, z, p \in X,\ d(x, y) + d(z, p) \leq \max\{d(x, z) + d(y, p), d(x, p) + d(y, z)\} + 2\delta$.

Definition 1.5. A geodesic metric space (X, d) is $\delta - hyperbolic$ if every geodesic triangle is $\delta - thin$, *i.e.*, given a geodesic triangle $\triangle xyz \subset X$, $\forall a \in [x, y], \exists b \in [x, z] \cup [z, y]$ such that $d(a, b) \leq \delta$. $[x, y]$ is the geodesic segment joint x, y.

Definition 1.3 is the original definition for δ-hyperbolic spaces from Gromov in [22], which depends on the notion of Gromov product and the Gromov product measures the failure of the triangle inequality to be an equality. This definition appears in almost every paper where δ-hyperbolic spaces are discussed, although one can provide a long list from our references we refer to [35] ,[9], [23], [24], [20] and [11]. The Gromov product enables one to define "convergence " at infinity and by this convergence the boundary of X, ∂X, can be defined. The metric on ∂X is the so called "visual metric" (see [9] and [11]). The advantage of Definition1.3 is that it facilitates the relationship between maps of δ-hyperbolic spaces and maps of their boundary [9], [26].

Definition 1.4 is a generalization of famous four-point property for which $\delta = 0$. Four-point property plays an important role in metric trees, for example, in [1], it is shown that a metric space is a metric tree if and only if it is complete, connected and satisfies the four-point property. However, it is also well known that a complete geodesic metric space X is a CAT(0) if and only if it satisfies four-point condition (see [11]). Furthermore, Godard in [20] proves that for a given metric space M, Lipschitz-free spaces $F(M)$ is isometric to a subspace of L_1 is equivalent to M satisfying four-point condition, thus equivalent to the fact that M isomerically embedds into a metric tree. The advantage of Definition 1.4 is that we can write out the inequality directly by distance of the metric space instead of by the Gromov product. In some cases if we construct a metric with the distance function having a particular form; it is easier to deal with distance inequality than Gromov product inequality. For example, in [23], [24] Ibragimov provides a method to construct a Gromov hyperbolic space by "hyperbolic filling" under a proper compact ultrametric space and such "filling" of a space contains points which are metric balls in original ultrametric space and is equipped with a distance function $h(A, B) = 2 \log \dfrac{diam\,(A \cup B)}{\sqrt{diam\,(A)\, diam\,(B)}}$. With the property of logarithmic function it is easy to show it is a $0 - hyperbolic$ space by using Definition1.4. For a similar application see [20].

Note that the Definition 1.5 of δ-hyperbolic spaces depends on geodecis triangles which needs the underlying space to be geodesic. Yet in [9] , Bonk and

Schramm show that any δ-hyperbolic space can be isometrically embedded into a geodesic δ-hyperbolic space. Thus one has the freedom of using Definition1.5.

Furthermore, recall that we call X is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$, sometimes δ is refereed as a *hyperbolicity constant* for X. Besides any tree being 0-hyperbolic, any space of finite diameter, δ, is δ-hyperbolic and the hyperbolic plane $\mathbb{H}^2$ is $(\frac{1}{2}\log 3)$-hyperbolic. In fact any simply connected Riemanian manifold with curvature bounded above by some negative constant $-\kappa^2 < 0$ is $(\frac{1}{2\kappa}\log 3)$-hyperbolic (see[11]).

2 Main Results

Theorem 2.1. *Definition 1.1 and Definition 1.2 of metric trees are equivalent.*

Proof. Suppose M is a $\mathbb{R}$-tree in the sense of Definition 1.1, and let $x, y \in M$. Then by Definition 1, there is a unique arc joining x and y which is isometric to an interval in $\mathbb{R}$. Hence it is a geodesic (i.e., metric) segment. So we may denote it by $[x, y]$. Thus we have defined a unique metric segment $[x, y]$ $(= [y, x])$ for each $x, y \in M$, so (i) holds.

To see that (ii) holds, suppose $[y, x] \cap [x, z] = \{x\}$. Then, $[y, x] \cup [x, z]$ is an arc joining y and z; and by Definition1.1 it must be isometric to a real line interval. Therefore it must be precisely the unique metric segment $[y, z]$. Now suppose M is a $\mathbb{R}$-tree in the sense of Definition 1.2, and let $x, y \in M$. Then $[x, y]$ is an arc joining x and y, and it is isometric with a real line interval. We must show that this is the only arc joining x and y.

Suppose A is an arc joining x and y, with $A \neq [x, y]$. By passing to a subarc, if necessary, we may without loss of generality, assume $A \cap [x, y] = \{x, y\}$. Let P be a nonexpansive projection of M onto $[x, y]$. Since P is continuous with $P(x) = x$ and $P(y) = y$, clearly there must exist $z_1, z_2 \in A \setminus \{x, y\}$ such that $P(z_1) \neq P(z_2)$. Let A_1 denote the subarc of A joining z_1 and z_2. Fix $z \in A_1$. If $u \in A_1$ satisfies $d(u, z) < d(z, P(z))$, then it must be the case that $P(u) = P(z)$. Here we use the fact that $[x, P(z)] \cap [P(z), z] = \{P(z)\}$; hence by (ii) $[x, z] = [x, P(z)] \cup [P(z), z]$. Therefore, there is an open neighborhood N_z of z such that $u \in N_z \cap A_1 \Rightarrow P(u) = P(z)$. The family $\{N_z\}_{z \in A}$ covers A_1, so, by compactness of A_1 there exist $\{z_1, \cdots, z_n\}$ in A_1 such that $A_1 \subset \bigcup_{i=1}^{n} N_{z_i}$. However, this implies $P(z_1) = P(z_2)$ which is a contradiction. Therefore, $A = [x, y]$, and since $[x, y]$ is isometric to an interval in $\mathbb{R}$, the conditions of Definition 1.1 are fulfilled. $\square$

Remark 2.2. In the above proof we used the fact that the closest point projection onto a closed metrically convex subset is noexpansive. Definition 1.2 is used in fixed point theory, mainly to investigate and see whether much of the known results for nonexpansive mappings remain valid in complete $CAT(0)$ spaces with asymptotic centre type of arguments used to overcome the lack of weak topology. For example it is shown that if C is a nonempty connected bounded open subset of a complete $CAT(0)$ space (M, d) and $T : \overline{C} \to M$ is nonexpansive, then either

1. T has a fixed point in $\overline{C}$, or

2. $0 < \inf\{d(x, T(x)) : \ x \in \partial C\}$.

Application of these to metrized graphs has led to "topological" proofs of graph theoretic results; for example refinement of the fixed edge theorem (see [27],[16], [28]). Definition1.1 used to construct T-theory and its relation to tight spans (see [15],) and best approximation in $\mathbb{R}$-trees (see [30]).

Theorem 2.3. *Definition 1.3, Definition 1.4 and Definition 1.5 of δ-hyperbolic spaces are equivalent.*

Proof. We suppose X be a geodesic Gromov $\delta - hyperbolic$ space below. We first show Definition1.3 implies Definition1.4. By Definition1.3, we have

$$d\left(x,p\right)+d\left(y,p\right)-d\left(x,y\right) \geq \min\left\{d\left(x,p\right)+d\left(z,p\right)-d\left(x,z\right), d\left(y,p\right)+d\left(z,p\right)-d\left(y,z\right)\right\}-2\delta.$$

Without loss of generality we can suppose

$$d\left(x,p\right)+d\left(z,p\right)-d\left(x,z\right) \geq d\left(y,p\right)+d\left(z,p\right)-d\left(y,z\right)$$

i.e.

$$d\left(x,p\right)+d\left(y,z\right) \geq d\left(y,p\right)+d\left(x,z\right).$$

So we have $d\left(x,p\right)+d\left(y,p\right)-d\left(x,y\right) \geq d\left(y,p\right)+d\left(z,p\right)-d\left(y,z\right)-2\delta$ or equivalently $\mathrm{d}(x,p)+d\left(y,z\right)+2\delta \geq d\left(z,p\right)+d\left(x,y\right).$
The same conclusion follows if we take

$$d\left(x,p\right)+d\left(z,p\right)-d\left(x,z\right) \leq d\left(y,p\right)+d\left(z,p\right)-d\left(y,z\right)$$

and we have $d\left(z,p\right)+d\left(x,y\right) \leq \max\left\{d\left(x,y\right)+d_{xz}, d_{xp}+d_{yz}\right\}+2\delta.$

To show Definition1.4 implies Definition 1.3,without loss of generality, we suppose

$$d\left(x,z\right)+d\left(y,p\right) \leq d\left(x,p\right)+d\left(y,z\right).$$

So, we have

$$d(x,y)+d(z,p) \leq d(x,p)+d(y,z)+2\delta.$$

Then

$$\begin{array}{rcl} d(y,p)+d(z,p)-d(y,z) & \leq & d(x,p)+d(z,p)-d(x,z) \\ d(y,p)+d(z,p)-d(y,z)-2\delta & \leq & d(x,p)+d(y,p)-d(x,y) \end{array}$$

and we get

$$d(x,p)+d(y,p)-d(x,y) \geq \min\{d(y,p)+d(z,p)-d(y,z), d(x,p)+d(z,p)-d(x,z)\}-2\delta.$$

To prove equivalence of Definition 1.3 and Definition 1.5, we need the following property:
For any geodesic triangle $\triangle xyz$ in metric space (M,d) we can find three points on each side denoted by a_x on $[y,z]$, a_y on $[x,z]$ and a_z on $[x,y]$ such that

$$\begin{array}{rcl} d(a_x,y) & = & d(a_z,y)=(x,z)_y \\ d(a_z,x) & = & d(a_y,x)=(y,z)_x \\ d(a_y,z) & = & d(a_x,z)=(x,y)_z . \end{array}$$

To show Definition1.5 implies Definition1.3, for any $x,y,p \in X$, we will show that for any $z \in X$ following holds:

$$(x,y)_p \geq \min\left\{(x,z)_p, (z,y)_p\right\} - 3\delta.$$

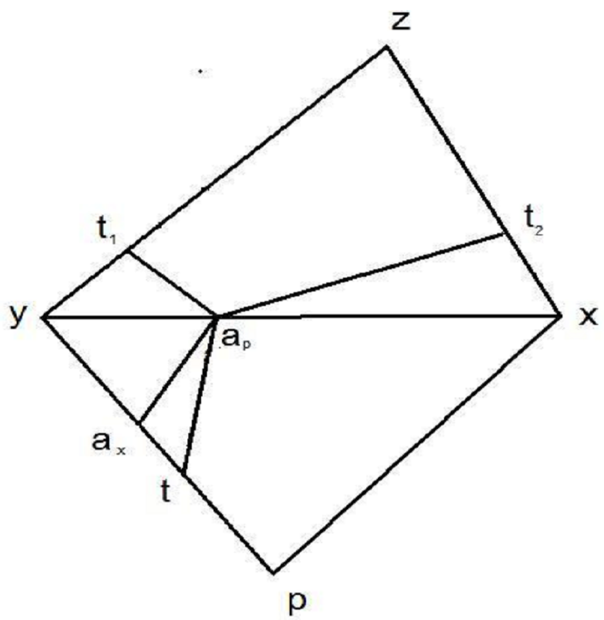

Figure 1

By the above stated property, in triangle $\triangle xyp$ we choose three points a_p, a_x and a_y in $[x, y], [p, y]$ and $[p, x]$ as shown in Figure 2 such that

$$\begin{aligned} d(y, a_p) &= (p, x)_y = d(y, a_x) \\ d(x, a_p) &= (p, y)_x = d(x, a_y) \\ d(p, a_x) &= (x, y)_p = d(p, a_y) \end{aligned}$$

Without loss of generality we assume $t \in [p, y]$ such that $d(a_p, t) \leq \delta$ and $d(t, y) > d(a_x, y)$, then in triangle $\triangle yta_p$,

$$d(t, y) < d(t, a_p) + d(a_p, y) = \delta + d(a_x, y)$$

so $d(t, a_x) < \delta$ and the same conclusion follows if we suppose $d(t, y) < d(a_x, y)$. Then for $\triangle a_p a_x t$,

$$d(a_p, a_x) < d(t, a_x) + d(a_p, t) < 2\delta.$$

For any $z \in X$, consider $\triangle xyz$ and choose $t_1 \in [y, z]$ and $t_2 \in [x, z]$ such that $d(a_p, t_1)$ and $d(a_p, t_2)$ are the shortest distances from a to $[y, z]$ and $[x, z]$, therefore

$$\min\{d(a, t_1), d(a, t_2)\} \leq \delta.$$

Then looking at triangles $\triangle pa_pt_1$ and $\triangle pa_pt_2$ we have

$$\begin{aligned}\min\{d(p,t_1),d(p,t_2)\} &\le \min\{d(a_p,t_1),d(a_p,t_2)\}+d(p,a_p)\\ &\le \delta+d(p,a_p)\le\delta+d(p,a_x)+d(a_p,a_x)\le 3\delta+(x,y)_p\,.\end{aligned}$$

Since

$$(y,z)_p=\frac{1}{2}\left(d(y,p)+d(z,p)-d(y,z)\right)=\frac{1}{2}\left(d(y,p)-d(y,t_1)+d(z,p)-d(z,t_1)\right)$$

by triangle inequality we have $(y,z)_p\le d(p,t_1)$ and similarly for $(x,z)_p\le d(p,t_2)\,.$ Then $\min\left((y,z)_p\,,(x,z)_p\right)\le\min\{d(p,t_1),d(p,t_2)\}\le 3\delta+(x,y)_p\,.$

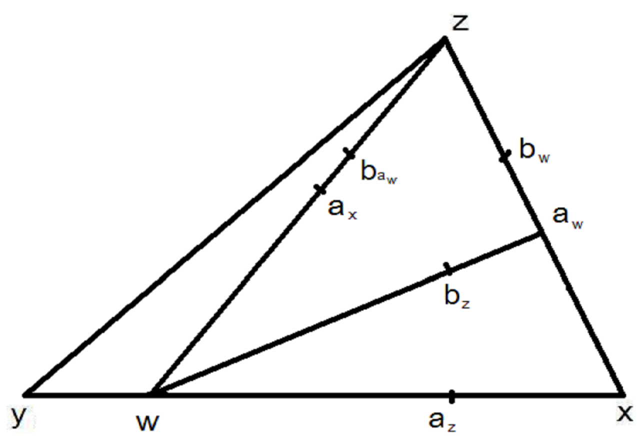

Figure 2

To show Definition 1.3 implies Definition 1.5, let $\triangle xyz$ be a geodesic triangle and $w\in[x,y]$, see Figure 2. Let $d(x,[y,z])$ denote the shortest distance from x to the side $[yz]$, without loss of generality we assume $(x,z)_w\ge(y,z)_w$. Then by Definition 1.3

$$(x,y)_w\ge\min\{(x,z)_w\,,(y,z)_w\}-\delta=(x,z)_w-\delta$$

which implies $\delta\ge(x,z)_w\,.$
Next we consider the triangle $\triangle xzw$ and find three points a_x,a_z and a_w on each side with the previous property. Then

$$\begin{aligned}d(w,a_x)&=(x,z)_w\\ d(w,a_w)&\ge d(w,[x,z])\,.\end{aligned}$$

Similarly, in $\triangle x a_w z$ one can find three points b_z, b_w and b_{a_w} on $[w, a_w], [a_w, z]$ and $[z, w]$, which satisfy the previous property and we have $d(w, b_{a_w}) = (z, a_w)_w$.
We assume $(z, a_w)_w < (x, a_w)_w$ then $d(w, a_x) < d(w, b_{a_w})$.
So,
$\delta \geq \min\{(z, a_w)_w, (x, a_w)_w\} - (x, z)_w = (z, a_w)_w - (x, z)_w = d(b_{a_w}, w) - d(a_x, w)$
$= d(a_x, b_{a_w}) = (a_w, z)_w - (x, z)_w$
$= \frac{1}{2}(d(a_w, w) + d(x, z) - d(a_w, z) - d(x, w))$
$= \frac{1}{2}(d(a_w, x) + d(w, b_z) + d(a_w, b_z) - d(w, a_z) - d(a_w, x))$
$= \frac{1}{2}(d(a_x, b_{a_w}) + d(a_w, b_w))$
which implies $d(a_x, b_{a_w}) = d(a_w, b_w)$.
Thus, $d(w, a_w) = d(w, b_z) + d(b_z, a_w) = d(w, b_{a_w}) + d(a_w, b_w) = d(w, a_x) + 2d(a_w, b_{a_w}) \leq (x, z)_w + 2\delta$.
Then $d(w, [x, z]) \leq d(w, a_w) \leq (x, z)_w + 2\delta \leq 3\delta$.

□

Remark 2.4. In [10] Bonk and Foertsch use the inequality (1.1) repeatedly to define a new space $AC_u(\kappa)$-space by introducing the notion of upper curvature bounds for Gromov hyperbolic spaces. This space is equivalent to a δ-hyperbolic space and furthermore it establishes a precise relationship between CAT(κ) spaces and δ-hyperbolic spaces. It is well known that any CAT(κ) space with negative κ is a δ-hyperbolic space for some δ. In [10] it is shown that a CAT(κ) space with negative κ is just an $AC_u(\kappa)$-space. Moreover, following the arguments in [10], Fournier, Ismail and Vigneron in [19] compute an approximate value for δ.

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